ON THE CONCORDANCE GENUS OF TOPOLOGICALLY SLICE KNOTS

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Abstract. The concordance genus of a knot K is the minimum three-genus of all knots smoothly concordant to K. We give a lower bound for the concordance genus of K coming from the knot Floer complex of K. As an application, we prove that there are topologically slice knots with 4-ball genus equal to one and arbitrarily large concordance genus.

1. Introduction

The *concordance genus* of a knot K, $g_c(K)$, is the minimum genus of all knots smoothly concordant to K. The concordance genus is bounded below by the 4-ball genus and above by the genus; that is,

$$g_4(K) \leq g_c(K) \leq g(K).$$

Concordance genus has been studied extensively by Livingston in [Liv04], where he shows that there are algebraically slice knots with 4-ball genus equal to one and arbitrarily large concordance genus.

In this paper, we give a lower bound for $g_c(K)$ coming from the knot Floer complex of K, and use this bound to give a family of topologically slice knots with smooth 4-ball genus equal to one and arbitrarily large concordance genus.

To a knot K in S^3, Ozsváth and Szabó [OS04b], and independently Rasmussen [Ras03], associate a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex, $CFK^\infty(K)$, whose filtered chain homotopy type is an invariant of K. Associated to this chain complex are several concordance invariants; in this paper, we focus on the invariant $\varepsilon(K)$, a $\{-1, 0, 1\}$-valued invariant defined in [Hom11], and to a lesser extent, the invariant $\tau(K)$, defined in [OS03].

We say that two $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complexes, C_1 and C_2, are *ε-equivalent* if

$$\varepsilon(C_1 \otimes C_2^*) = 0,$$

where C^* denotes the dual of C. We say that two knots, K_1 and K_2, are *ε-equivalent* if their knot Floer complexes are ε-equivalent, that is, if

$$\varepsilon\big(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*\big) = 0.$$

As seen in the following theorem, ε-equivalence is closely related to concordance:

Theorem 1 ([Hom11])**.** *If two knots are concordant, then they are ε-equivalent.*

We define the *breadth* of a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex C, $b(C)$, to be

$$b(C) = \max\{j \mid H_*(C(0, j)) \neq 0\},$$

where $C(i, j)$ denotes the (i, j)-graded summand of the associated graded complex. Recall from [OS04a, Theorem 1.2] that

$$g(K) = b(CFK^\infty(K)).$$

The invariant $\gamma(K)$ is defined to be the minimum breadth of all filtered chain complexes ε-equivalent to $CFK^\infty(K)$:

$$\gamma(K) = \min\{b(C) \mid \varepsilon(CFK^\infty(K) \otimes C^*) = 0\}.$$

Theorem 2. *The invariant $\gamma(K)$ gives a lower bound on the smooth concordance genus of K; that is,*

$$g_c(K) \geq \gamma(K).$$

At this first glance, this may seem like an intractable invariant, as the set of chain complexes ε-equivalent to $CFK^\infty(K)$ is infinite. However, in many situations, there are tractable numerical invariants associated the the ε-equivalence class of K giving lower bounds for $\gamma(K)$, and hence also for $g_c(K)$. In this next theorem, we use these bounds to prove a result concerning the concordance genus of a family of topologically slice knots.

Let D denote the (positive, untwisted) Whitehead double of the right-handed trefoil, and let $K_{p,q}$ denote the (p,q)-cable of K, where p indicates the longitudinal winding and q the meridional winding. We write $-K$ to denote the reverse of the mirror of K.

Theorem 3. *Let $K_p = D_{p,1} \# -D_{p-1,1}$. Then K_p is topologically slice with $g_4(K_p) = 1$ and $g_c(K_p) \geq p$.*

Recall that in [Liv04, Theorem 1.5], Livingston constructs algebraically slice knots with 4-ball genus equal to one and arbitrarily large concordance genus. However, his proof relies on Casson-Gordon invariants, and so his examples are not topologically slice. He also remarks on the inherent challenge in bounding the concordance genus: one must show that the given knot is not concordant to any knot in the infinite family of knots with genus less than a given N. The invariant γ can help significantly in this regard. Moreover, the invariant γ can bound the concordance genus of topologically slice knots, while the techniques of [Liv04] cannot.

Organization. In Section 2, we recall the necessary properties of Heegaard Floer homology and knot Floer homology, and use them to prove Theorem 2. In Section 3, we apply those results to give a family of topologically slice knots with 4-ball genus one and arbitrarily large concordance genus.

We work with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout. Unless otherwise stated, we work in the smooth category.

Acknowledgements. I would like to thank Chuck Livingston for hosting me in Bloomington, Indiana, where he suggested this problem to me, and Peter Horn for many helpful conversations.

2. Bounding the concordance genus

We recall the basic definitions of knot Floer homology, assuming that the reader is familiar with these invariants; for an expository overview, we suggest [OS06]. In this paper, we concern ourselves primarily with the algebraic properties of the invariant.

To a knot $K \subset S^3$, Ozsváth-Szabó [OS04b], and independently Rasmussen [Ras03], associate $CFK^\infty(K)$, a $\mathbb{Z}$-graded, $\mathbb{Z}$-filtered freely generated chain complex over the ring $\mathbb{F}[U, U^{-1}]$, where U is a formal variable. The filtered chain homotopy type of $CFK^\infty(K)$ is an invariant of the knot K. The differential does not decrease the U-exponent, and the U-exponent (more precisely, the negative of the U-exponent) induces a second $\mathbb{Z}$-filtration, giving $CFK^\infty(K)$ the structure of a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex. The ordering on $\mathbb{Z} \oplus \mathbb{Z}$ is given by $(i,j) \leq (i',j')$ if $i \leq i'$ and $j \leq j'$.

This chain complex is freely generated over $\mathbb{F}[U, U^{-1}]$ by tuples of intersection points in a doubly pointed Heegaard diagram for S^3 compatible with the knot K. Each generator x comes with a homological, or Maslov grading, $M(x)$, and an Alexander filtration, $A(x)$. The differential, ∂, decreases the Maslov grading by one, and respects the Alexander filtration; that is,

$$M(\partial x) = M(x) - 1 \qquad \text{and} \qquad A(\partial x) \leq A(x).$$

Multiplication by U shifts the Maslov grading by two and decreases the Alexander filtration:

$$M(U \cdot x) = M(x) - 2 \qquad \text{and} \qquad A(U \cdot x) = A(x) - 1.$$

It is often convenient to graphically represent this complex in the (i, j)-plane, where the i-axis corresponds to $-(U\text{-exponent})$, and the j-axis corresponds to the Alexander filtration. The Maslov grading is suppressed from this picture. A generator x is placed at $(0, A(x))$, and a element of the form $U^n \cdot x$ is placed at $(-n, A(x) - n)$.

We depict the differential using arrows. Given a basis $\{x_i\}$ for $CFK^\infty(K)$, we can express the boundary of an element, say x_0, as a linear combination of basis elements, with coefficients in $\mathbb{F}[U]$. If the coefficient of x_i in ∂x_0 is non-zero (i.e., it is U^{n_i} for some non-negative integer n_i), then we say that $U^{n_i} \cdot x_i$ is in the boundary of x_0, and place an arrow from x_0 to $U^{n_i} \cdot x_i$. The differential respects the $\mathbb{Z} \oplus \mathbb{Z}$-filtration, and so the arrows will necessarily point (non-strictly) to the left and down.

Given a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex C and $S \subset \mathbb{Z} \oplus \mathbb{Z}$, we write $C\{S\}$ to denote the set of elements in the plane whose (i, j)-coordinates are in S together with the arrows between them. If S has the property that $(i, j) \in S$ implies that $(i', j') \in S$ for all $(i', j') \leq (i, j)$, then $C\{S\}$ is a subcomplex of C. We write $C(i, j)$ to denote the subquotient complex with coordinates (i, j), that is, $C\{(i, j)\}$.

The $\mathbb{Z}$-filtered complex $\widehat{CFK}(K)$ is the subquotient complex consisting of the j-axis, i.e., $C\{i \leq 0\}/C\{i < 0\}$. The homology of the associated graded object of $\widehat{CFK}(K)$ is $\widehat{HFK}(K)$. The groups $\widehat{HFK}(K)$ can themselves be viewed as a chain complex, with the differential induced by the higher order, i.e., non-filtration preserving, differentials on $\widehat{CFK}(K)$. Moreover, $\widehat{HFK}(K)$ is a basis over $\mathbb{F}[U, U^{-1}]$ for $CFK^\infty(K)$. Choosing $\widehat{HFK}(K)$ as a basis for $CFK^\infty(K)$ has the advantage that it is *reduced*; that is, the differential strictly lowers the filtration. Graphically, this means that each arrow will point strictly downward or to the left (or both).

We have the following chain homotopy equivalences [OS04b, Theorem 7.1 and Section 3.5]:

$$\begin{aligned} CFK^\infty(K_1 \# K_2) &\simeq CFK^\infty(K_1) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^\infty(K_2) \\ CFK^\infty(-K) &\simeq CFK^\infty(K)^* \end{aligned}$$

where $CFK^\infty(K)^*$ denotes the dual of $CFK^\infty(K)$, i.e., $\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(CFK^\infty(K), \mathbb{F}[U, U^{-1}])$.

To fully exploit the richness of this invariant, it is helpful to study certain induced maps on homology. For example, the Ozsváth-Szabó concordance invariant τ is defined in [OS03] to be

$$\tau(K) = \min\{s \mid \iota : C\{i = 0, j \leq s\} \to C\{i = 0\} \text{ induces a non-trivial map on homology}\},$$

where ι is the natural inclusion of chain complexes. Note that $H_*(C\{i = 0\}) \cong \widehat{HF}(S^3) \cong \mathbb{F}$. The invariant $\tau(K)$ provides a lower bound on the 4-ball genus of K, and gives a surjective homomorphism from the smooth concordance group to the integers [OS03].

More recently, the $\{-1, 0, 1\}$-valued concordance invariant $\varepsilon(K)$ has been defined in [Hom11]. To define ε, it is helpful to consider the map on homology, F_*, induced by the chain map

$$F : C\{i = 0\} \to C\{\min(i, j - \tau) = 0\}$$

where $\tau = \tau(K)$, and the chain map consists of quotienting by $C\{i = 0, j < \tau\}$ followed by the inclusion of $C\{i = 0, j \geq \tau\}$ into $C\{\min(i, j - \tau) = 0\}$. Similarly, we consider the map G_*, induced by

$$G : C\{\max(i, j - \tau) = 0\} \to C\{i = 0\},$$

the composition of quotienting by $C\{i < 0, j = \tau\}$ and including $C\{i = 0, j \leq \tau\}$ into $C\{i = 0\}$.

Definition 2.1. *The invariant ε is defined in terms of F_* and G_* as follows:*

- $\varepsilon(K) = 1$ *if F_* is trivial (in which case G_* is necessarily non-trivial).*
- $\varepsilon(K) = -1$ *if G_* is trivial (in which case F_* is necessarily non-trivial).*
- $\varepsilon(K) = 0$ *if F_* and G_* are both non-trivial.*

See [Hom11, Section 3] for details.

Further invariants are defined in [Hom11, Section 6]. Suppose $\varepsilon(K) = 1$, and consider the map on homology H_s induced by the chain map

$$C\{i = 0\} \to C\{\min(i, j - \tau) = 0, i \leq s\},$$

where s is a non-negative integer, and the map consists of quotienting by $C\{i = 0, j < \tau\}$, followed by inclusion. When s is sufficiently large, the map H_s is trivial since $\varepsilon(K) = 1$, while when $s = 0$, it is not difficult to see that the map H_s is non-trivial. Thus, one can define

$$a_1 = \min\{s \mid H_s \text{ is trivial}\}.$$

Going even further, consider the map on homology $H_{a_1,s}$ induced by

$$C\{i = 0\} \to C\big\{\{\min(i, j - \tau) = 0, i \leq a_1\} \cup \{i = a_1, \tau - s \leq j < \tau\}\big\},$$

where the map consists of quotienting by $C\{i = 0, j < \tau\}$, followed by inclusion. Define

$$a_2 = \min\{s \mid H_{a_1,s} \text{ is non-trivial}\}.$$

The set $\{s \mid H_{a_1,s}$ is non-trivial$\}$ may be empty – there is no reason why the map $H_{a_1,s}$ must be non-trivial for any s – in which case the invariant $a_2(K)$ is undefined.

Two knots K_1 and K_2 are *ε-equivalent* if

$$\varepsilon(K_1 \# -K_2) = 0.$$

Concordant knots are ε-equivalent, and a_1 and a_2 are invariants of ε-equivalence [Hom11, Theorem 2 and Lemma 6.1]

Proof of Theorem 2. The proof that $\gamma(K)$ gives a lower bound on concordance genus is an immediate consequence of the definition of $\gamma(K)$, as follows. By Theorem 1, any two concordant knots are ε-equivalent. Since $g(K) = b(CFK^\infty(K))$ by [OS04a, Theorem 1.2] and

$$\gamma(K) = \min\{b(C) \mid C \text{ is } \varepsilon\text{-equivalent to } CFK^\infty(K)\},$$

it follows immediately that

$$g_c(K) \geq \gamma(K).$$

□

At times, it may be difficult to compute $\gamma(K)$ directly, but we can bound it using the invariants $\tau(K)$, $a_1(K)$, and $a_2(K)$.

Lemma 2.2. *Suppose that $\varepsilon(K) = 1$, and $a_2(K)$ is defined. Then*

$$\gamma(K) \geq |\tau(K) - a_1(K) - a_2(K)|.$$

Proof. From the basis found in [Hom11, Lemma 6.2] and the fact that τ, a_1, and a_2 are invariants of ε-equivalence, it follows that

$$H_*\Big(C\big(0,\tau(K)-a_1(K)-a_2(K)\big)\Big)\neq 0,$$

for any complex C that is ε-equivalent to $CFK^\infty(K)$. Using the various symmetry properties of $CFK^\infty(K)$ [OS04b, Section 3.5], it follows that

$$H_*\big(C(0,|\tau(K)-a_1(K)-a_2(K)|\big)\neq 0,$$

as well. This implies that $b(C)\geq|\tau(K)-a_1(K)-a_2(K)|$ for any C that is ε-equivalent to $CFK^\infty(K)$, giving the desired bound. □

Remark 2.3. Recall that $|\tau(K)|\leq g_4(K)\leq g_c(K)$. When $\varepsilon(K)=1$ and $\tau(K)\leq 0$, the invariant $\gamma(K)$ will give a better bound on $g_c(K)$ than $\tau(K)$; that is, $\gamma(K)>|\tau(K)|$.

3. The knots $D_{p,1}\#-D_{p-1,1}$

Let D denote the (positive, untwisted) Whitehead double of the right-handed trefoil. Let $K_{p,q}$ denote the (p,q)-cable of K, where p indicates the longitudinal winding and q the meridional winding. We will study various properties of the family of knots

$$D_{p,1}\#-D_{p-1,1},\quad p>1.$$

Since the Alexander polynomial of D is equal to one, by Freedman [Fre82] D is topologically slice. Hence the $(p,1)$-cable of D is topologically concordant to the underlying pattern torus knot, which is unknotted. It follows that the knot $D_{p,1}\#-D_{p-1,1}$ is topologically slice.

In the following lemma, we will show that these knots are never smoothly slice.

Lemma 3.1. *The smooth 4-ball genus of the knot $D_{p,1}\#-D_{p-1,1}$ is equal to one.*

Proof. A genus p Seifert surface for $D_{p,1}$ can be built from p parallel copies of a genus one Seifert surface for D, and $p-1$ bands connecting them. Likewise, we may build a genus $p-1$ Seifert surface for $-D_{p-1,1}$. Connecting these two Seifert surfaces together with a band yields a genus $2p-1$ Seifert surface F for $D_{p,1}\#-D_{p-1,1}$. The slice knot $D_{p-1,1}\#-D_{p-1,1}$ sits on F, and furthermore bounds a subsurface of genus $2p-2$. We may perform surgery on F in B^4 along $D_{p-1,1}\#-D_{p-1,1}$, yielding a genus one slice surface for $D_{p,1}\#-D_{p-1,1}$.

By [Hed07], $\tau(D)=1$, and by [Hed09, Theorem 1.2] (cf. [Hom12, Theorem 1]), it follows that $\tau(D_{p,1})=p$. Therefore, $\tau(D_{p,1}\#-D_{p-1,1})=1$, which is a lower bound on the 4-ball genus of the knot [OS03]. Since this bound can be realized, it follows that $g_4(D_{p,1}\#-D_{p-1,1})=1$. □

To bound the concordance genus of $K_p=D_{p,1}\#-D_{p-1,1}$, we consider its knot Floer complex. We do this using the tools of [Hom11] together with the bordered Floer homology package of Lipshitz, Ozsváth, and Thurston [LOT08], as applied to cables by Petkova [Pet09].

The knot D is ε-equivalent to the $(2,3)$-torus knot $T_{2,3}$ [Hom11, Lemma 6.12]. Moreover, if two knots are ε-equivalent, then so are their satellites [Hom11, Proposition 4]. Therefore, to understand D and its satellites from the perspective of ε-equivalence, we may instead work with $T_{2,3}$ and its satellites. The advantage of this is the knot Floer complex of $T_{2,3}$ is simpler to work with from a computational perspective. It has rank three, and is homologically thin, meaning that $\widehat{HFK}(T_{2,3})$ is supported on a single diagonal with respect to its bigrading.

Cables of homologically thin knots are studied by Petkova in [Pet09], where she describes $\widehat{HFK}(K_{p,pn+1})$ for any homologically thin knot K, in terms of the Alexander polynomial of K,

$\tau(K)$, p, and n. The proof of her main result relies on bordered Floer homology, and the same techniques can be used to determine the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(K_{p,pn+1})$.

Since $T_{2,3}$ is homologically thin, we may use Theorem 1 of [Pet09] to compute the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(T_{2,3;p,1})$, from which we can determine certain information about $CFK^\infty(T_{2,3;p,1})$, which is ε-equivalent to $CFK^\infty(D_{p,1})$. More precisely, this information will be the invariants a_1 and a_2, which will determine the bounds on concordance genus necessary for Theorem 3.

Lemma 3.2. *Let D denote the (positive, untwisted) Whitehead double of the right-handed trefoil, and $D_{p,1}$ its $(p,1)$-cable, $p > 1$. Then $a_1(D_{p,1}) = 1$ and $a_2(D_{p,1}) = p$.*

Proof. By the remarks above, the knot $T_{2,3;p,q}$ is ε-equivalent to $D_{p,1}$, so we will study $CFK^\infty(T_{2,3;p,1})$ instead of $CFK^\infty(D_{p,1})$.

We use [Pet09, Theorem 1] to determine $\widehat{HFK}(T_{2,3;p,1})$. The rank of $\widehat{HFK}(T_{2,3;p,1})$ is $6p-5$. The generators and their gradings are summarized in Table 1. (Note that our gradings differ from those in [Pet09] in the following ways: our Alexander grading A is the negative of Petkova's, and our Maslov grading M is Petkova's N. By the symmetry properties of knot Floer homology [OS04b, Section 3.5], this change in grading does not affect $\widehat{HFK}$. Our preference in grading conventions is chosen so that we may compute the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(K)$ rather than the U-module $HFK^-(K)$.)

Generator	(M,A)	$M+2p-2A$	
au_1	$(0,p)$	0	
b_1v_1	$(-1,p-1)$	1	
$b_1\mu_1$	$(-2,-1)$	$2p$	
b_iv_2	$(-2i-1,-i)$	$2p-1$	$1 \le i \le p-2$
$b_{i+1}\mu_1$	$(-2i-2,-i-1)$	$2p$	$1 \le i \le p-2$
$b_{p-1}v_2$	$(-2p+1,-p+1)$	$2p-1$	
b_pv_2	$(-2p,-p)$	$2p$	
b_iv_1	$(-1,-i+p)$	$-1+2i$	$2 \le i \le p-1$
$b_{2p-1-i}v_1$	$(-2,0)$	$2p-2$	$2 \le i \le p-1$
$b_i\mu_2$	$(0,-i+p)$	$-2i$	$1 \le i \le p-1$
$b_{2p-1-i}\mu_2$	$(-1,0)$	$2p-1$	$1 \le i \le p-1$

TABLE 1. $\widehat{HFK}(T_{2,3;p,1})$

Furthermore, we may compute the higher order differentials on $\widehat{HFK}$, i.e., the homotopy type of the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}$. (This corresponds to the U-exponents in [Pet09, Section 4].) To do so, we use the well-known "edge reduction" procedure for chain complexes; see, for example, [Lev10, Section 2.6]. Following the computation in [Pet09], there is a summand of $\widehat{CFK}(T_{2,3;p,1})$ consisting of the generators

$$au_2, \quad au_3, \quad b_1v_1, \quad b_1\mu_1, \quad b_{2p-2}v_1, \quad b_{2p-2}\mu_1$$

with the nonzero differentials

$$\begin{aligned}\partial(au_2) &= au_3 + b_1\mu_1 + b_{2p-2}v_1 \\ \partial(b_1v_1) &= b_{2p-2}v_1 \\ \partial(b_1\mu_1) &= b_{2p-2}\mu_1 \\ \partial(au_3) &= b_{2p-2}\mu_1,\end{aligned}$$

where au_2 and $b_{2p-2}v_1$ both have Alexander grading zero, and au_3 and $b_{2p-2}\mu_1$ both have Alexander grading $-p$. Thus, we "cancel" the edge the edge between au_2 and $b_{2p-2}v_1$, and the edge between au_3 and $b_{2p-2}\mu_1$, which introduces an edge between b_1v_1 and $b_1\mu_1$. The summand now consists of

$$b_1v_1, \quad b_1\mu_1$$

with the differential

$$\partial(b_1v_1) = b_1\mu_1.$$

Similarly, when $p \geq 3$, there is a summand of $\widehat{CFK}(T_{2,3;p,1})$ consisting of the generators

$$b_iv_2, \quad b_{2p-i-1}v_2, \quad b_{i+1}\mu_1, \quad b_{2p-i-2}\mu_1$$

for $1 \leq i \leq p-2$, with the following nonzero differentials

$$\begin{aligned}\partial(b_iv_2) &= b_{2p-i-1}v_2 + b_{i+1}\mu_1 \\ \partial(b_{2p-i-1}v_2) &= b_{2p-i-2}\mu_1 \\ \partial(b_{i+1}\mu_1) &= b_{2p-i-2}\mu_1,\end{aligned}$$

where $b_{2p-i-1}v_2$ and $b_{2p-i-2}\mu_1$ have the same Alexander grading. After canceling the edge between $b_{2p-i-1}v_2$ and $b_{2p-i-2}\mu_1$, we reduce the summand to

$$b_iv_2, \quad b_{i+1}\mu_1$$

with the nonzero differential

$$\partial(b_iv_2) = b_{i+1}\mu_1.$$

After applying the edge reduction procedure, the nonzero higher differentials on $\widehat{HFK}(T_{2,3;p,1})$ are

$$\begin{aligned}\partial(b_1v_1) &= b_1\mu_1 & & \\ \partial(b_iv_2) &= b_{i+1}\mu_1, & & 1 \leq i \leq p-2 \\ \partial(b_{p-1}v_2) &= b_pv_2 & & \\ \partial(b_iv_1) &= b_{2p-1-i}v_1, & & 2 \leq i \leq p-1 \\ \partial(b_i\mu_2) &= b_{2p-1-i}\mu_2, & & 1 \leq i \leq p-1.\end{aligned}$$

The basis for $\widehat{CFK}$ given by $\widehat{HFK}$ in this example has a particularly simple form. In the language of [LOT08, Definition 11.25], it is *simplified*; that is, there is at most one arrow starting or ending at each basis element.

By [Hom12, Theorem 2], we know that $\varepsilon(T_{2,3;p,1}) = 1$. We also know that au_1 is a generator of the total homology $H_*(\widehat{CFK}(T_{2,3;p,1}))$. We will now find a basis satisfying the conditions in Lemma 6.2 of [Hom11], and in doing so, will determine the values of $a_1(T_{2,3;p,1})$ and $a_2(T_{2,3;p,1})$. In order to accomplish this, we will need to find an element whose horizontal boundary in $CFK^\infty(T_{2,3;p,1})$ is au_1.

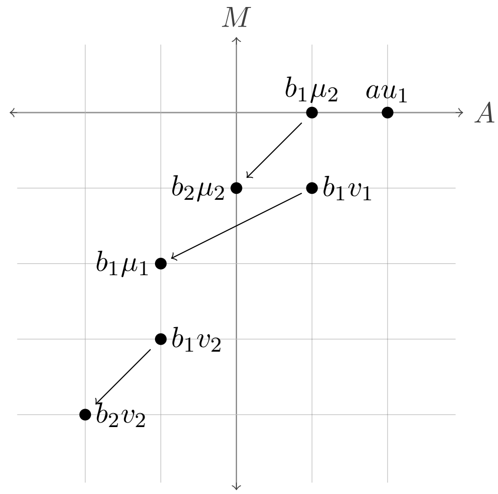

FIGURE 1. $\widehat{CFK}(T_{2,3;2,1})$, where the horizontal axis represents the Alexander grading and the vertical axis represents the homological, or Maslov, grading.

We will view the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex $CFK^\infty(K)$ in the (i,j)-plane. The complex $\widehat{CFK}(K)$ can be viewed as the subquotient complex of $CFK^\infty(K)$ consisting of elements with i-coordinate equal to zero. We place a generator x at the lattice point $(0, A(x))$, where $A(x)$ denotes the Alexander grading of x. For example, the generator b_1v_1 has coordinates $(0, p-1)$. The Maslov grading is suppressed from the picture, although we will still keep track of it.

The knot Floer complex $CFK^\infty(K)$ is generated over $\mathbb{F}[U, U^{-1}]$ by $\widehat{CFK}(K)$. Recall that the horizontal i-axis represents $-U$-exponent; that is, an element $U^n \cdot x$ has (i,j)-coordinates $(-n, A(x)-n)$, and Maslov grading $M(x)-2n$.

We would like to find an element with j-coordinate equal to $\tau(T_{2,3;p,1})$ whose horizontal boundary is equal to au_1. In particular, we would like to find an element with j-coordinate equal to p, i-coordinate greater than zero, and Maslov grading one, which is one more than the Maslov grading of au_1. To find the elements with j-coordinate equal to p, we view the appropriate U-translates of elements in $\widehat{CFK}(K)$. More specifically, given a generator x of $\widehat{CFK}(K)$, the translate $U^{A(x)-p}\cdot x$ will be in the p^{th}-row, with

$$A(U^{A(x)-p}\cdot x) = p \qquad \text{and} \qquad M(U^{A(x)-p}\cdot x) = M(x) + 2p - 2A(x).$$

By considering the gradings in the third column of Table 1, which are the Maslov gradings of the elements in the j^{th}-row, we see that the only element in that row with Maslov grading one is $U^{-1}\cdot b_1v_1$.

Since $\varepsilon(T_{2,3;p,1}) = 1$, there must be an element whose horizontal boundary is au_1, and by grading considerations, we have shown that that element must be $U^{-1}\cdot b_1v_1$. The vertical boundary of $U^{-1}\cdot(b_1v_1)$ is $U^{-1}\cdot b_1\mu_1$, and

$$A(U^{-1}\cdot b_1v_1) = A(U^{-1}\cdot b_1\mu_1) + p.$$

It follows that

$$a_1(T_{2,3;p,1}) = 1 \qquad \text{and} \qquad a_2(T_{2,3;p,1}) = p,$$

and since $T_{2,3;p,1}$ and $D_{p,1}$ are ε-equivalent, the result follows. $\square$

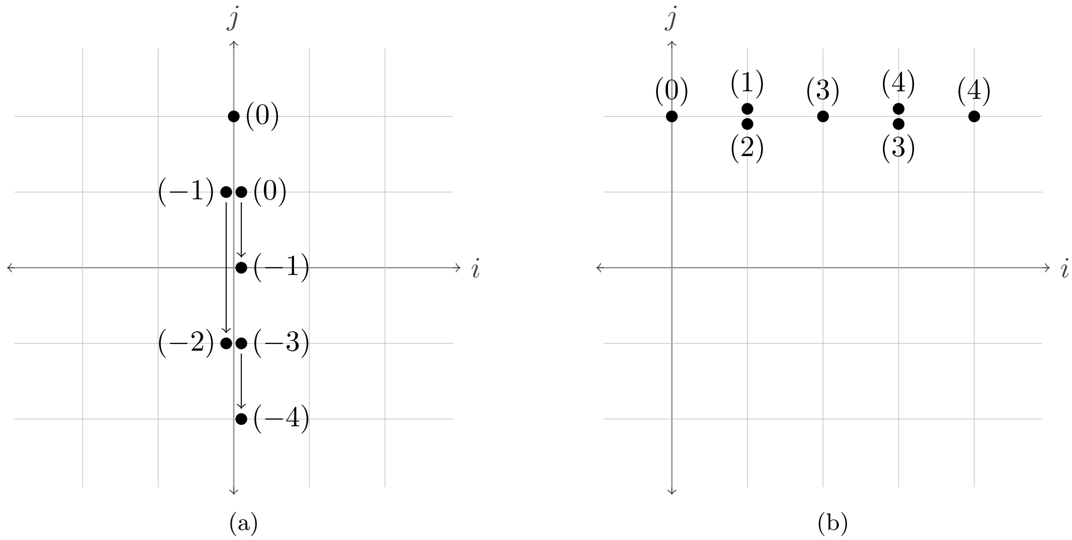

FIGURE 2. Left, the complex $\widehat{CFK}(T_{2,3;2,1})$, in the (i,j)-plane, with the vertical differentials. Right, the U-translates of the complex $\widehat{CFK}(T_{2,3;2,1})$ to the $j = 2$ row. The numbers in parentheses indicate the Maslov gradings of the generators.

We are now ready to prove Theorem 3, giving an infinite family of topologically slice knots with 4-ball genus one and arbitrarily large concordance genus.

Proof of Theorem 3. By Lemma 3.2,

$$a_1(D_{p,1}) = 1 \qquad \text{and} \qquad a_2(D_{p,1}) = p.$$

In the proof of [Hom11, Lemma 6.4], it is shown that given knots J and K, if $a_1(J) = a_1(K)$ and $a_2(J) > a_2(K)$, then

$$a_1(J\# -K) = a_1(J) \qquad \text{and} \qquad a_2(J\# -K) = a_2(J).$$

In particular,

$$a_1(D_{p,1}\# -D_{p-1,1}) = 1 \qquad \text{and} \qquad a_2(D_{p,1}\# -D_{p-1,1}) = p.$$

In the beginning of this section, it was observed that the knots $D_{p,1}\# -D_{p-1,1}$ are topologically slice, and in Lemma 3.1, we saw that $g_4(D_{p,1}\# -D_{p-1,1}) = 1$.

By Theorem 2 and Lemma 2.2, we see that

$$\begin{aligned} g_c(D_{p,1}\# -D_{p-1,1}) &\geq |\tau(D_{p,1}\# -D_{p-1,1}) - a_1(D_{p,1}\# -D_{p-1,1}) - a_2(D_{p,1}\# -D_{p-1,1})| \\ &= |1-1-p| \\ &= p, \end{aligned}$$

completing the proof of the theorem. □

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